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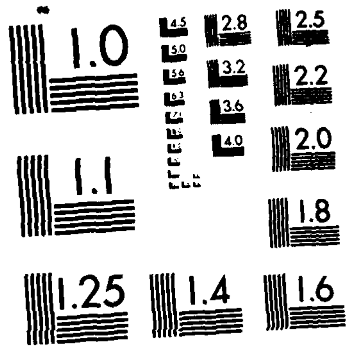

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TECHNICAL REPORT BRL-TR-2702

ON THE EXISTENCE OF LI-YORKE POINTS IN THE THEORY OF CHAOS

Nam P. Bhatia
Walter O. Egerland

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TECHNICAL REPORT BRL-TR-2702	2. GOVT ACCESSION NO. ADA 164691	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On the Existence of Li-Yorke Points in the Theory of Chaos.		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Nam P. Bhatia Walter O. Egerland		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Ballistic Research Laboratory ATTN: SLCBR-SE Aberdeen Proving Ground, MD 21005-5066		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS RDT&E 1L162618AH43
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Ballistic Research Laboratory ATTN: SLCBR-DD-T Aberdeen Proving Ground, MD 21005-5066		12. REPORT DATE January 1986
		13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The first author is Professor of Mathematics at the University of Maryland, Baltimore County. The results were established under the U.S. Army Summer Faculty Research and Engineering Program and will be published in the Journal of Nonlinear Analysis, Pergamon Press LTD., Oxford, England.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Theory of Chaos
Li-Yorke Inequalities and Points
Non-periodic Conditions for Chaos

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report establishes an addendum to the Li-Yorke theorem, lists the complete set of Li-Yorke inequalities, and provides in the Equivalence theorem non-periodic conditions equivalent to the known periodic conditions that guarantee chaos.

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I. INTRODUCTION

Li and Yorke[1] introduced in their fundamental paper the term "chaotic" for a class of self-mappings of an interval. The real function f is chaotic if (a) there are points of arbitrarily large periods and (b) there is an uncountable set S such that for every $x_0, y_0 \in S$, $x_0 \neq y_0$, $\lim_{n\to\infty} \sup | f^n(x_0) - f^n(y_0) | > 0$ and $\lim_{n\to\infty} \inf | f^n(x_0) - f^n(y_0) | = 0$. Following the Li-Yorke result that "period three implies chaos," many authors worked on periodic conditions that allow the same conclusion. The best known of these conditions is "period $p \neq 2^n$ implies chaos." Such investigations are summarized in Targonski's monograph.[2]

Li and Yorke also introduced four-point inequalities satisfied by a point and its three successors with respect to the given function f. They showed that these imply the existence of a three-period and hence chaos. Our investigations in this paper show that the Li-Yorke inequalities play a fundamental role in the theory of chaos. We, therefore, formalize the notion of a Li-Yorke point and obtain non-periodic conditions equivalent to the known periodic conditions that guarantee chaos. The importance of the results resides in the fact that the set of Li-Yorke points is always an open set, whereas the number of periodic points of a given period $p \neq 2^n$ is in general countable.

II. DEFINITIONS AND NOTATIONS

Let $f: R \to R$ be continuous. If $x_0 \in R$, the orbit of x_0 under f is defined as the set $\{x : x = f^n(x_0), n = 0, 1, \ldots\}$, where, for every positive integer n, f^n is the n-th iterate of f and $f^0(x_0) = x_0$. We shall write $x_n := f^n(x_0)$ for a given $x_0 \in R$ and call $x_1, x_2 \ldots$ the successors of x_0. A pre-orbit of a given $x_0 \in R$ is any (finite or infinite) sequence $x_0, x_{-1}, x_{-2} \ldots$ such that $f(x_{-n}) = x_{-(n-1)}$ for all n for which x_{-n} is defined. The points $x_{-1}, x_{-2} \ldots$ in any such sequence are called predecessors of x_0. A point x_0 is called critical if $f(x_0) = x_0$, i.e., a critical point of f is a fixed point of f. A periodic point x_0 of period $p > 1$ (p a positive integer) is a point for which the relations $f^p(x_0) = x_0$, $f^k(x_0) \neq x_0$, $1 \leq k < p$, hold.

The following fundamental results are now well-known.

<u>Theorem</u> (Sarkovskii).[3] For $m, n = 0, 1, \ldots$ consider the total ordering of the positive integers:

$$3 < 5 < 7 < \ldots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 \ldots < 2^n \cdot 3 < 2^n \cdot 5 < 2^n \cdot 7$$
$$< \ldots < 2^m < 2^{m-1} < \ldots < 2^2 < 2 < 1.$$

If a continuous mapping $f : R \to R$ has a periodic point of period p, then it also has a periodic point of period q for every $q > p$ (in the above total order).

<u>Theorem</u> (Li-Yorke): Let $f : R \to R$ be continuous. If there is a point $x_0 \in R$ such that either $x_3 < x_0 < x_1 < x_2$ or $x_3 > x_0 > x_1 > x_2$, then f has a point of period three.

[1] T.-Y. Li and J. A. Yorke, "Period three implies Chaos.," <u>Amer. Math. Monthly</u> 82 (1975), pp. 985-992.
[2] Gyorgy Targonski, "Topics in Iteration Theory," <u>Studia Mathematica, Skript 6,</u> Vandenhoek and Ruprecht, Gottingen and Zurich (1981).
[3] A. N. Sarkovskii, "Coexistence of Cycles of a Continuous Map of a Line into Itself," <u>Ukrain. Mat. Zh. 16</u> (1964), pp. 61-71.

Theorem (Li-Yorke): Let $f : R \to R$ be continuous. If there is a point $x_0 \in R$ such that either $x_3 < x_0 < x_1 < x_2$ or $x_3 > x_0 > x_1 > x_2$, then f has a point of period three. Furthermore, if f has a three periodic point, there exists an uncountable set $S \subset R$ such that for every $x_0, y_0 \in S$, $x_0 \neq y_0$,

$$\lim_{n \to \infty} \sup | x_n - y_n | > 0$$

and

$$\lim_{n \to \infty} \inf | x_n - y_n | = 0.$$

Definition. A point $x_0 \in R$ is called a Li-Yorke point of $f : R \to R$ if x_0 satisfies a Li-Yorke inequality

$$x_3 < x_0 < x_1 < x_2$$

or

$$x_3 > x_0 > x_1 > x_2.$$

III. AN EXAMPLE

The following example shows that the existence of a three periodic point does not guarantee the existence of a Li-Yorke point. The quadratic mapping $g(y) = ay^2 + 2by + c$, $a \neq 0$, b, c, real constants, may be brought in the form $f(x) = x^2 - r$ by setting $y = a^{-1}(x - b)$, $g = a^{-1}(f - b)$, and $r = b^2 - b - ac$. f has a three periodic orbit for $r = 7/4$, but no point $x_0 \in R$ is a Li-Yorke point for $r \leq 7/4$.

IV. THE RELATIONSHIP BETWEEN LI-YORKE POINTS AND POINTS OF PERIOD THREE

We apply frequently the following crossing property of real continuous functions.

Lemma. Let $f(x)$ be continuous on $[a,b]$. If either

$$f(a) \geq a, \quad f(b) \leq b$$

or

$$f(a) \leq a, \quad f(b) \geq b,$$

then there exists $x \in [a, b]$ such that $x = f(x)$. In particular, the existence of such an $x \in [a, b]$ follows from the inclusions $[a,b] \subset [f(a), f(b)]$ or $[a,b] \supset [f(a), f(b)]$.

We shall denote the k-th iterate of x_0 under the function f^m by x_k^m, $k = 0, 1 \ldots$ Thus $x_k^m = (f^m)^k(x_0) = x_{mk}$, and, in particular, $x_0^m = x_k^0 = x_0$ for all nonnegative integers k and m.

Theorem 1. If f has a three periodic point, then f^2 has a Li-Yorke point.

Proof. Let y_0 be a three periodic point of f. Thus $y_0 = y_3$ and $y_i \neq y_k$, $0 \le i < k \le 3$. We can assume without loss of generality that $y_0 < y_1 < y_2$. Since $f([y_1, y_2]) \supset [y_0, y_2] \supset [y_1, y_2]$, there is a critical point $c_0 \in (y_1, y_2)$. Hence

$$y_0 < y_1 < c_0 < y_2.$$

Since $f(y_0) = y_1 < c_0$ and $f(y_1) = y_2 > c_0$, we have a $c_{-1} \in (y_0, y_1)$, and since $f(y_2) = y_0 < c_{-1}$ and $f(c_0) = c_0 > c_{-1}$, there is a $c_{-2} \in (c_0, y_2)$, thus yielding the inequality

$$y_0 < c_{-1} < y_1 < c_0 < c_{-2} < y_2 .$$

Similar reasoning establishes points c_{-j}, $j = 3, 4, 5, 6$, and the inequality

$$y_0 < c_{-1} < y_1 < c_{-3} < c_{-5} < c_0 < c_{-6} < c_{-4} < c_{-2} < y_2 .$$

If we identify c_{-6} with x_0, we obtain $c_{-6} = x_0$, $c_{-4} = x_1^2$, $c_{-2} = x_2^2$, $c_0 = x_3^2$ and the inequality $x_3^2 < x_0 < x_1^2 < x_2^2$. This completes the proof.

Theorem 2. If f has a Li-Yorke point x_0, f has at least two distinct three periodic orbits.

Proof. Let $S = \{x_0 : x_3 < x_0 < x_1 < x_2 \text{ or } x_2 < x_1 < x_0 < x_3\}$. Then S is non-empty by hypothesis and open because f is continuous. Each component of S is an open interval. If $I = (a_0, b_0)$ is a component of S, we will show that $-\infty < a_0 < b_0 < \infty$ and that both a_0 and b_0 are three periodic points in different orbits. Fix $x_0 \in I$ and assume that $x_3 < x_0 < x_1 < x_2$. Then the same inequality holds for all $x_0 \in I$. Assuming first that $-\infty < a_0 < b_0 < \infty$, we note that a_0 and b_0 are limit points of points x_0 for which $x_3 < x_0 < x_1 < x_2$. Hence by continuity of f we must have the inequalities $a_3 \le a_0 \le a_1 \le a_2$ and $b_3 \le b_0 \le b_1 \le b_2$ with at least one equality sign holding in each of them. We claim that $a_3 = a_0 < a_1 < a_2$ and $b_3 = b_0 < b_1 < b_2$. Noting that $f(x_1) = x_2 > x_1$ and $f(x_2) = x_3 < x_2$, it follows that there is a critical point $c_0 \in (x_1, x_2)$, so that $x_3 < x_0 < x_1 < c_0 < x_2$. Since $f(x_0) = x_1 < c_0$ and $f(x_1) = x_2 > c_0$ imply the existence of $c_{-1} \in (x_0, x_1)$ with $f(c_{-1}) = c_0$, we have $x_3 < x_0 < c_{-1} < x_1 < c_0 < x_2$. As $a_0 < x_0$ and $f(a_0) = a_1 = a_0 < c_{-1}$, $f(x_0) = x_1 > c_{-1}$, we have $c_{-2} \in (a_0, x_0) \subset I$ with $f(c_{-2}) = c_{-1}$. But $c_{-2} < c_{-1} < c_0 = c_1$ yields by relabelling a $y_0 = c_{-2} \in I$ such that $y_0 < y_1 < y_2 = y_3$, a contradiction. Hence $a_0 < a_1$. If now $a_1 = a_2$, we have $a_0 < a_1 = a_2 = a_3$ which contradicts $a_3 \le a_0 \le a_1 \le a_2$. Therefore, we must have $a_1 < a_2$, and hence $a_3 = a_0 < a_1 < a_2$, i.e., a_0 is a three periodic point. Similarly, b_0 is a three periodic point for which $b_3 = b_0 < b_1 < b_2$ holds. Since $a_0 < b_0$, a_0 is not in the orbit of b_0, and so the orbits of a_0 and b_0 are distinct. It remains to show that $a_0 \neq -\infty$ and $b_0 \neq \infty$. Since $x_0 < b_0$ and $(x_0, b_0) \subset I$ and $c_{-1} \notin S \supset I$, we must have $x_0 < b_0 < c_{-1}$ as $x_0 < c_{-1}$. This shows that $b_0 < \infty$. To see that $a_0 > -\infty$, we note that $x_3 \in f^2[c_{-1}, c_0]$, i.e., x_3 lies in a compact set. If $A = \inf f^2[c_{-1}, c_0]$, then for $x_0 < A$ we conclude $x_0 < x_3$ which contradicts $x_0 \in I$ with $a_0 = -\infty$. This completes the proof.

The theorem extends the Li-Yorke theorem and may be illustrated by the example $f(x) = x^2 - 2$. The point $x_0 = \sqrt{2}$ is a Li-Yorke point of f and $(2 \cos \frac{2}{7}\pi, 2 \cos \frac{4}{7}\pi, 2 \cos \frac{8}{7}\pi)$, $(2 \cos \frac{2}{9}\pi, 2 \cos \frac{4}{9}\pi, 2 \cos \frac{8}{9}\pi)$ are two distinct three periodic orbits.

The following theorem extends the Li-Yorke theorem in another direction. It establishes equivalent companion inequalities to the Li-Yorke inequalities.

Theorem 3. The sets of points $A = \{x_0 : x_3 < x_0 < x_1 < x_2\}$, $B = \{x_0 : x_1 < x_2 < x_0 < x_3\}$, and $C = \{x_0 : x_2 < x_3 < x_0 < x_1\}$ are either all empty or all non-empty. The same statement holds for the sets in which all inequalities are reversed.

Proof. Let $A \neq \phi$. Then for some $x_0 \in R$ we have $x_3 < x_0 < x_1 < x_2$. Since $f(x_1) = x_2 > x_0$ and $f(x_2) = x_3 < x_0$, there is $x_{-1} \in (x_1, x_2)$ such that $x_0 < x_1 < x_{-1} < x_2$. Upon relabelling $y_0 = x_{-1}$, we have $y_1 < y_2 < y_0 < y_3$, so that $B \neq \phi$, i.e., $A \neq \phi$ implies $B \neq \phi$. Let $B \neq \phi$. Then, if $x_1 < x_2 < x_0 < x_3$, it follows from $f(x_0) = x_1 < x_0$ and $f(x_2) = x_3 > x_2$ that there is a critical point $c_0 \in (x_2, x_0)$ such that $x_1 < x_2 < c_0 < x_0 < x_3$. Since $f(c_0) = c_0 < x_0$ and $f(x_2) = x_3 > x_0$, there is $x_{-1} \in (x_2, x_0)$, so that $x_1 < x_2 < x_{-1} < x_0$. Letting $y_0 = x_{-1}$, we have $y_2 < y_3 < y_0 < y_1$. Thus $B \neq \phi$ implies $C \neq \phi$. Finally, assume $C \neq \phi$. If for some $x_0 \in R$ $x_2 < x_3 < x_0 < x_1$, then there exists $x_{-1} \in (x_2, x_0)$ since $f(x_2) = x_3 < x_0$ and $f(x_0) = x_1 > x_0$. From $x_2 < x_{-1} < x_0 < x_1$ follows, setting $x_{-1} = y_0$, $y_3 < y_0 < y_1 < y_2$, and hence that $C \neq \phi$ implies $A \neq \phi$. This completes the proof of the theorem.

Examples show that other four-point inequalities between x_0, x_1, x_2, x_3 do not assure the existence of three-periodic orbits. Therefore, the theorem lists the complete set of Li-Yorke four-point inequalities.

Theorem 4 (Equivalence Theorem). The following statements are equivalent.

(a) f has a point of period $p \neq 2^n$.

(b) There exists $m \geq 1$ such that f^m has a point of period three.

(c) f^{2m} has Li-Yorke points.

(d) f^{2m} has at least two three-periodic orbits.

Proof. If f has a point of period $p \neq 2^n$, then f has a point of period $3 \cdot 2^k$ for some $k \geq 0$ by the theorem of Sarkovskii. Such a point is a three-periodic point of f^m, $m = 2^k$. Hence (a) implies (b). (b) implies (c) by Theorem 1, and (c) implies (d) by Theorem 2. Now let x_0 be a three-periodic point of f^q, $q = 2m$, $x_0 = f^{3q}(x_0)$, $x_0 \neq f^q(x_0)$, $x_0 \neq f^{2q}(x_0)$. x_0 is thus a periodic point of f of minimal period N, $2 \leq N \leq 3q$. This implies that $x_0 = f^{kN}(x_0)$ for every $k \geq 1$ and $x_0 \neq f^s(x_0)$ if $s \neq kN$, $k = 1, 2, \ldots$ Hence $3q = k' N$ for some $k' \geq 1$. This shows that either 3 divides k' or 3 divides N. If 3 divides k', then $q = sN$ and consequently $f^q(x_0) = x_0 = f^{sN}(x_0)$, a contradiction. Hence 3 divides N. But then $N \neq 2^n$ for every $n \geq 0$. Therefore, f has period $N \neq 2^n$ and (d) implies (a). This completes the proof of the Equivalence Theorem.

V. REFERENCES

1. T.-Y. Li and J. A. Yorke, "Period three implies Chaos.," Amer. Math. Monthly 82 (1975), pp. 985-992.

2. Gyorgy Targonski, "Topics in Iteration Theory," Studia Mathematica, Skript 6, Vandenhoek and Ruprecht, Gottingen and Zurich (1981).

3. A. N. Sarkovskii, "Coexistence of Cycles of a Continuous Map of a Line into Itself," Ukrain. Mat. Zh. 16 (1964), pp. 61-71.

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